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ESTIMATION OF THE LARGER MEAN

by

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Estimation of the larger of two normal means μ_1 and μ_2 is studied. Two new estimators are studied for this problem, the maximum probability estimator and the iterated bias elimination estimators. If $|\mu_1 - \mu_2|$ is not close to zero (as evidenced by a strongly consistent estimator), a maximum probability estimator is seen to be best as regards both bias and mean-squared error.

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Estimation of the Larger Mean

by

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Abstract

In the present paper estimation of the larger of two normal means is studied. Two new estimators are added to the class of possible estimators of the larger normal mean, namely, the maximum probability estimator and the iterated bias elimination estimators. If the magnitude of the difference between the two population means is not close to zero (as evidenced by its strongly consistent estimator, the magnitude of the difference between the two sample means) a suitably chosen maximum probability estimator is seen to be best as regards both bias and mean-squared error.

1. Introduction

Let $X_{i1},\ldots,X_{in}$ be a random sample of size n from a normal population with mean μ_i and variance σ^2, $i = 1,\ldots,k$ (≥ 2). Let $\mu_{[1]} \leq \cdots \leq \mu_{[k]}$ be the ordered unknown means and suppose σ^2 is known. The estimation analog of the well-known ranking and selection problem [see Bechhofer, Kiefer, and Sobel (1968)] has been the topic of inquiry by Alam (1967), Blumenthal (1975, 1976), Blumenthal and Cohen (1968), Dudewicz (1971. 1972, 1973, 1976) and others. Once the decision has been made as to which of the k populations has the largest mean, it is natural to ask, "How large is this largest mean?" For two examples of application of estimation of the larger normal mean, the reader is referred to Blumenthal and Cohen (1968).

In this paper we study this problem and attempt to enlarge the class of estimators of the larger mean. In section 2 we investigate, mostly numerically, the behavior of a class of estimators called Maximum Probability Estimators (MPE's) introduced by Weiss and Wolfowitz (1967). Dudewicz (1973) first gave the MPE's of the ranked means. In section 3 a new class of estimators called Iterated Bias Elimination Estimators (IBEE's) are introduced and investigated. A comparison along the lines of Blumenthal and Cohen (1968) is made in section 4.

2. Maximum Probability Estimators

Definition (Weiss and Wolfowitz). Let Θ be a closed region in the m-dimensional Euclidean space $\mathbb{R}^m$, $\Theta \subset \bar{\Theta}$, where $\bar{\Theta}$ is a closed region such that every finite boundary point of Θ is an inner point of $\bar{\Theta}$. For each n let $X(n)$ denote the (finite) vector of random variables of which the estimator is to be a function. Let $K_n(x,\theta)$ be the density of $X(n)$ with respect to a sigma-finite measure. Let R be a fixed bounded set in $\mathbb{R}^m$ and let $k(n) = (k_1(n),\ldots,k_m(n))$ be a sequence of numbers such that $k_i(n) \to \infty$ $(n \to \infty)$ for each i. Let $d = (d_1,\ldots,d_m)$ and $d - R/k(n) = \{(z_1,\ldots,z_m) \in \bar{\Theta}: d_i - y_i/k_i(n) = z_i,\ i = 1,\ldots,m;\ (y_1,\ldots,y_m) \in R\}$. Then Z_n is an MPE with respect to R and $k(n)$ if $Z_n(x)$ equals a $d \in \bar{\Theta}$ such that

$$\int_{d-R/k(n)} K_n(x,\theta)d\theta = \sup_{t \in \bar{\Theta}} \int_{t-R/k(n)} K_n(x,\theta)d\theta . \tag{2.1}$$

Now let $\bar{X}_1,\ldots,\bar{X}_k$ denote the sample means. Let in the above definition

$$X(n) = (\bar{X}_{[1]},\dots,\bar{X}_{[k]}),$$
$$\bar{\Theta} = \mathbb{R}^k,$$
$$\Theta = \{\underline{\mu} = (\mu_1,\dots,\mu_k) \in \mathbb{R}^k : \ \mu_1 = \mu_{[1]},\dots,\mu_k = \mu_{[k]}\}$$

and

$$K_n(x,\theta) = f_{\bar{X}_{[1]},\dots,\bar{X}_{[k]}}(x_1,\dots,x_k;\mu)$$

with respect to Lebesgue measure on $\mathbb{R}^k$ where $\bar{X}_{[1]} \le \dots \le \bar{X}_{[k]}$ are ordered sample means. (Note that $K_n(x,\theta) > 0$ iff $x_1 \le \dots \le x_k$.) Let

$$k(n) = (\sqrt{n}/\sigma,\dots,\sqrt{n}/\sigma)$$

and

$$R = (-r_1/2,r_1/2) \times (-r_2/2,r_2/2) \times \dots \times (-r_k/2,r_k/2) ,$$

$r_1,\dots,r_k$ being positive real numbers. Define

$$d\text{-}R/k(n) = \left\{(\mu_1,\dots,\mu_k) \in \mathbb{R}^k : \ d_i - \frac{\sigma r_i}{2\sqrt{n}} \le \mu_i \le d_i + \frac{\sigma r_i}{2\sqrt{n}} , \ i = 1,\dots,k\right\} .$$

We know that

$$f_{\bar{X}_{[1]},\dots,\bar{X}_{[k]}}(x_1,\dots,x_k;\mu) = (\sqrt{n}/\sigma)^k \sum_{\beta \in S_k} \prod_{i=1}^{k} \phi\left(\frac{x_{\beta(i)} - \mu_i}{\sigma/\sqrt{n}}\right) ,$$

where S_k is the set of permutations on integers $1,\dots,k$ and $\phi(\cdot)$ denotes the standard normal density function. With $d_{\beta(i)} = \sqrt{n}(x_{\beta(i)} - x_i)/\sigma$ and $t_i = x_i + a_i\sigma/\sqrt{n}$ we find from the definition that $t = (t_1,\dots,t_k)$ is an MPE for $\mu = (\mu_{[1]},\dots,\mu_{[k]})$ if $a = (a_1,\dots,a_k)$ are chosen so as to achieve

$$\sup_{a} \sum_{\beta \in S_k} \prod_{i=1}^{k} \left\{\Phi(d_{\beta(i)}-a_i + r_i/2) - \Phi(d_{\beta(i)}-a_i-r_i/2)\right\} \qquad (2.2)$$

where $\Phi(\cdot)$ denotes the distribution function of a standard normal variable.

For values of k up to about 20 and using the observed values of d_i's we can find $a_1,\ldots,a_k$ by utilizing some function maximization (minimization) algorithm such as that of Nelder and Mead (1965).

In the rest of this section we investigate numerically the MPE's of $(\mu_{[1]},\mu_{[2]})$, that is, for the case $k = 2$. For the values of the difference between the larger and the smaller sample means considered, after a preliminary analysis, we found that we could treat $r_1, r_2 = 0.5$, 1.5, and 5.0 as small, medium, and large respectively for the study. Thus we have nine pairs of (r_1,r_2) to look at. We found numerically that for $d \leq \min(r_1,r_2)$, $a_1 = -a_2 = d/2$ maximize

$$g(a_1,a_2) = \sum_{\beta \in S_2} \prod_{i=1}^{2} \left\{\Phi(d_{\beta(i)} - a_i + r_i/2) - \Phi(d_{\beta(i)} - a_i - r_i/2)\right\}$$

$$= \left\{\Phi(a_1 + r_1/2) - \Phi(a_1 - r_1/2)\right\} \left\{\Phi(a_2 + r_2/2) - \Phi(a_2 - r_2/2)\right\}$$

$$+ \left\{\Phi(a_1 - d + r_1/2) - \Phi(a_1 - d - r_1/2)\right\} \left\{\Phi(a_2 + d + r_2/2) - \Phi(a_2 + d - r_2/2)\right\},$$

where $d = \sqrt{n}(x_2 - x_1)/\sigma$. This observation was checked with r_1, r_2 changing in steps of 0.1 from 0.1 to 3.5. Redefining t_1 and t_2 as $t_1 = x_1 + a_1\sigma/\sqrt{n}$ and $t_2 = x_2 - a_2\sigma/\sqrt{n}$ we can write $g(a_1,a_2)$ as

$$g(a_1,a_2) = \left\{\Phi(a_1 + r_1/2) - \Phi(a_1 - r_1/2)\right\} \left\{\Phi(a_2 + r_2/2) - \Phi(a_2 - r_2/2)\right\}$$

$$+ \left\{\Phi(a_1 - d + r_1/2) - \Phi(a_1 - d - r_1/2)\right\} \left\{\Phi(a_2 - d + r_2/2) - \Phi(a_2 - d - r_2/2)\right\} \quad (2.3)$$

and now $a_1 = a_2 = d/2$ maximize $g(a_1,a_2)$ for $d \leq \min(r_1,r_2)$. Table 1 gives the values of (a_1,a_2) for the above nine pairs of (r_1,r_2) for $d = 0.1(0.1)3.5$. We tabulate for only six (r_1,r_2) pairs because of the subscript symmetry in (2.3) and hence the other three columns can be deduced from this table itself. For these calculations the Nelder-Mead (1965) simplex method was utilized. For each of the nine (r_1,r_2) pairs and each value of d, it was verified that $g(a_1^0 \pm 10^{-7}, a_2^0 \pm 10^{-7}) < g(a_1^0, a_2^0)$ where (a_1^0,a_2^0) is the calculated value. Thus the values reported are correct to seven decimal places.

Bias. Now without loss of generality, we consider the MPE's of $\mu_{[2]}$ only. For given r_1,r_2, the MPE of $\mu_{[2]}$ is given by

$$t_2 = t_2(r_1,r_2) = \bar{X}_{[2]} - \tau\, a_2(Z, r_1, r_2)$$

where $\tau = \sigma/\sqrt{n}$ and $Z = (\bar{X}_{[2]} - \bar{X}_{[1]})/2$. Therefore, the bias is

$$\begin{aligned} B(t_2) &= B(t_2, \omega, \tau) \\ &= B(\bar{X}_{[2]}) - \tau \int_0^{\infty} a_2(z, r_1,r_2) f(z, \omega, \tau)dz \end{aligned} \tag{2.4}$$

where $B(\bar{X}_{[2]})$ denotes the bias of $\bar{X}_{[2]}$ as an estimator of $\mu_{[2]}$ and

$$f(z, \omega, \tau) = \frac{\sqrt{2}}{\tau}\left\{\phi(\sqrt{2}(z-\omega)/\tau) + \phi(\sqrt{2}(z+\omega)/\tau)\right\}, \; z > 0 \tag{2.5}$$

is the density of Z with $\omega = (\mu_{[2]} - \mu_{[1]})/2$. Also from Blumenthal and Cohen (1968) we have

$$B(\bar{X}_{[2]}) = \frac{\tau}{\sqrt{\pi}} e^{-\omega^2/\tau^2} - 2\omega\Phi(-\sqrt{2}\,\omega/\tau). \tag{2.6}$$

From (2.4), (2.5), and (2.6) it is clear that $B(t_2, \omega, \tau) = \tau\, B(t_2, \omega/\tau, 1)$. Therefore, we take $\tau = 1$ in the following calculations and the values reported are in units of τ. Values of $B(\bar{X}_{[2]})$ are given in Blumenthal and Cohen (1968); we independently verified these values.

Now, from Dudewicz (1973) we know that $0 < a_2 < 2Z$. Also $\phi(\sqrt{2}\,(z+\omega)) < \phi(\sqrt{2}(z-\omega))$. Therefore, if we approximate $\int_0^{\infty} a_2(z, r_1, r_2)f(z, \omega)dz$ by $\int_0^{M} a_2(z, r_1, r_2)f(z, \omega)dz$, the error due to this truncation is

$$E_T = \int_M^{\infty} a_2(z, r_1, r_2) f(z, \omega) dz$$

$$\leq 4\sqrt{2} \int_M^{\infty} z \, \phi(\sqrt{2}(z-\omega)) dz.$$

In order to bound E_T by ϵ, it suffices to find an $M = M_\epsilon(\omega)$ such that

$$4\sqrt{2} \int_{M_\epsilon(\omega)}^{\infty} z \, \phi(\sqrt{2}(z-\omega)) dz \leq \epsilon.$$

Prior to the numerical evaluation of the integral $I_M = \int_0^{M_\epsilon(\omega)} a_2(z, r_1, r_2) f(z, \omega) dz$ a study of the function $g(z, r_1, r_2, \omega) = a_2(z, r_1, r_2) f(z, \omega)$ revealed that this function has a sharp peak (possibly a non-differentiable point) in the interval $[0, M_\epsilon(\omega)]$. Three typical functions are graphed in Figure 1. This fact is an indication that one should evaluate the integral I_M in two parts, namely, in intervals $[0,a]$ and $[a,b]$, where $a = a(\omega)$ is the point such that $a_2(z, r_1, r_2) = z$ for $z \leq a$ [note that a turns out to be the same point where $a_2(z, r_1, r_2)$ starts decreasing] and $b = b(\omega)$ is such that $a_2(z, r_1, r_2) \leq \epsilon$ for $z \geq b$. It was found that in each case considered $b \leq M_\epsilon(\omega)$. Since for $z \geq b$, $a_2(z, r_1, r_2) \leq \epsilon$.

$$\int_b^{\infty} a_2(z, r_1, r_2) f(z, \omega) dz \leq \epsilon.$$

Hence we used Gaussian quadrature formula [see Stroud and Secrest (1966)]

$$\int_0^a a_2(z, r_1, r_2) f(z, \omega) dz + \int_a^b a_2(z, r_1, r_2) f(z, \omega) dz$$

to approximate

$$\int_0^{\infty} a_2(z, r_1, r_2) f(z, \omega) dz.$$

To control the error due to quadrature, for each (r_1, r_2) pair each of the integrals

$$I_1 = \int_0^a a_2(z, r_1, r_2) f(z, \omega) dz$$

and

$$I_2 = \int_a^b a_2(z, r_1, r_2) f(z, \omega) dz$$

was evaluated in 3, 5, 7, or 9 subintervals using 64 and 128 point Gaussian quadrature formulas. The criterion used to stop subdividing each subinterval was that the two values of the integral I_i obtained using 64 and 128 points respectively for a subdivision of the subinterval differ by no more than ϵ^*, $i = 1,2$. Thus our approximation involves an error due to quadrature plus an error due to truncation which is bounded above by $\epsilon + \epsilon^*$. We used $\epsilon = \epsilon^* = 10^{-7}$. Thus we have

$$B(t_2) \cong B(\overline{X}_{[2]}) - I_1 - I_2$$

<u>Mean-squared error (MSE).</u> We have

$$\begin{aligned} MSE(t_2) &= MSE(t_2, \omega, \tau) \\ &= MSE(\overline{X}_{[2]}) + E\,[\tau^2 a_2^2(Z, r_1, r_2) - 2\tau(Z-\omega) a_2(Z, r_1, r_2)] \\ &= MSE(\overline{X}_{[2]}) - \tau^2 \int_0^\infty \left\{ 2\left(\frac{Z}{\tau} - \frac{\omega}{\tau}\right) a_2(Z, r_1, r_2) - a_2^2(Z, r_1, r_2) \right\} f(z, \omega, \tau) dz. \end{aligned} \qquad (2.7)$$

Also

$$MSE(\overline{X}_{[2]}) = \tau^2 + 4\omega^2 \Phi(-\sqrt{2}\,\omega/\tau) - (2\omega\tau/\sqrt{\pi}) e^{-\omega^2/\tau^2} \qquad (2.8)$$

has been tabulated by Blumenthal and Cohen (1968); we independently verified these values. Again, it is clear that $MSE(t_2, \omega, \tau) = MSE(t_2, \frac{\omega}{\tau}, 1)$. All the calculations run parallel to those for the bias except that new bounds are found to approximate the integral in (2.7).

Tables 2 and 3 below give the bias and MSE respectively of $t_2(r_1, r_2)$. We see that smaller values of r_1, r_2 give rise to smaller |bias| and smaller MSE.

3. Iterated Bias Elimination Estimators

Consider the case of k = 2 normal populations with means μ_1 and μ_2 and a common known variance σ^2. Let $X^{*(1)} = \bar{X}_{[2]}$ and let $B^{*(1)}(\omega)$ denote the bias of $X^{*(1)}$ as an estimator of $\mu_{[2]}$. Let $\hat{B}^{*(1)}(\omega)$ denote a consistent estimator of $B^{*(1)}(\omega)$. Define $X^{*(2)} = X^{*(1)} - \hat{B}^{*(1)}(\omega)$ which has a bias of $B^{*(2)}(\omega)$ as an estimator of $\mu_{[2]}$. Let $\hat{B}^{*(2)}(\omega)$ denote a consistent estimator of $B^{*(2)}(\omega)$. Continuing the process, let $X^{*(m)} = X^{*(m-1)} - \hat{B}^{*(m-1)}(\omega)$, m = 2,3,... . In this section we investigate the class of estimators $\{X^{*(m)};\ m = 1,2,\ldots\}$ of $\mu_{[2]}$.

Let $C^{*(0)}(\omega) = \frac{\tau}{\sqrt{\pi}} e^{-\omega^2/\tau^2} - 2\omega\Phi(-\sqrt{2}\,\omega/\tau)$, and $C^{*(m)}(\omega) = \frac{\sqrt{m+1}}{\sqrt{\pi}}\tau\, e^{-\omega^2/\tau^2(m+1)} - 2\omega\Phi(-\sqrt{2}\,\omega/\sqrt{m+1}\,\tau)$, m = 1,2,... . We estimate ω (which appears in $B^{*(m)}(\omega)$ through the continuous function $C^{*(m-1)}(\omega)$, m = 1,2,...) by its strongly consistent estimator Z. That is, $\hat{B}^{*(m)}(\omega) = B^{*(m)}(Z)$.

<u>Definition.</u> $X^{*(m)}$ is called the m-th IBEE of $\mu_{[2]}$, m = 1,2,... .

<u>Lemma.</u> For m = 1,2,...

$$E\, C^{*(m-1)}(Z) = C^{*(m)}(\omega) - C^{*(0)}(\omega).$$

Proof. The proof is straightforward involving just the routine evaluation of integrals of type

$$\int_0^\infty e^{a_1 z^2} \phi(b_1 z + c_1)dz + \int_0^\infty z\, \Phi(a_2 z)\phi(b_2 z + c_2)dz.$$

Theorem. For $m = 1,2,\ldots$

$$B^{*(m)}(\omega) = \sum_{r=0}^{m-1} (-1)^r \binom{m}{r+1} C^{*(r)}(\omega) .$$

Proof. From Blumenthal and Cohen (1968) we know that the theorem is true for $m = 1$. Suppose it is true for $m = k$. Then, by definition

$$X^{*(k+1)} = X^{*(k)} - \sum_{r=0}^{k-1} (-1)^r \binom{k}{r+1} C^{*(r)}(Z).$$

Therefore,

$$B^{*(k+1)}(\omega) = B^{*(k)}(\omega) - \sum_{r=0}^{k-1} (-1)^r \binom{k}{r+1} E\, C^{*(r)}(Z)$$

$$= \sum_{r=0}^{k-1} (-1)^r \binom{k}{r+1} C^{*(r)}(\omega) - \sum_{r=0}^{k-1} (-1)^r \binom{k}{r+1} \{C^{*(r+1)}(\omega) - C^{*(0)}(\omega)\}$$

$$= \sum_{r=0}^{k} (-1)^r \binom{k+1}{r+1} C^{*(r)}(\omega).$$

Thus, the theorem is true for $m = k+1$. The proof now follows by induction.

Corollary. For $m = 2,3,\ldots$

$$X^{*(m)} = X^{*(1)} - \sum_{r=0}^{m-2} (-1)^r \binom{m}{r+2} C^{*(r)}(Z).$$

A numerical study as to the behavior of $B^{*(m)}(\omega)$ as m increases shows that for smaller values of ω $B^{*(m)}(\omega)$ decreases as m is increased. The behavior is reversed for larger values of ω. Table 4 gives for $m = 1,2,\ldots,30$

$$b(m) = \max_{\omega = 0.0(0.1)5.0} |B^{*(m)}(w)|$$

which reveals that for values of m and ω considered $X^{*(18)}$ is the minimax |bias| IBEE.

4. Comparison of Estimators of $\mu_{[2]}$.

In this section we make comparisons between the "best" estimators of Blumenthal and Cohen (1968), the "best" MPE and the "minimax" |bias| IBEE as regards their bias and MSE. From tables and graphs of Blumenthal and Cohen we find that with respect to the bias the estimator

$$\delta_H(1) = \begin{cases} (\bar{X}_1 + \bar{X}_2)/2 & \text{if } z < \tau \\ \bar{X}_{[2]} & \text{if } z \geq \tau \end{cases}$$

seems to be the "best" among all the estimators considered and with respect to MSE, the estimator

$$\delta_P = \frac{\bar{X}_1 + \bar{X}_2}{2} + 2Z[\Phi(Z/\sqrt{2}\,\tau) - \tfrac{1}{2}] + \frac{\tau}{\sqrt{\pi}} e^{-Z^2/\tau^2}$$

seems to have an advantage over the other estimators. From section 2 we know that among the nine MPE's considered $t_2(0.5,0.5)$ is the 'best' both in case of bias and MSE. In Tables 5 and 6 we give the bias and MSE respectively of the four estimators $\delta_H(1)$, δ_P, $t_2(0.5,0.5)$, and $X^{*(18)}$ and compare them graphically in Figures 2 and 3. (Note that the MSE($X^{*(18)}$) was calculated using Monte Carlo techniques.)

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Table 1: Maximum Probability Estimators

d	$r_1=0.5$, $r_2=0.5$		$r_1=0.5$, $r_2=1.5$		$r_1=0.5$, $r_2=5.0$		$r_1=1.5$, $r_2=1.5$		$r_1=1.5$, $r_2=5.0$		$r_1=5.0$, $r_2=5.0$	
	a_1	a_2	a_1	a_2	a_1	a_2	a_1	a_2	a_1	a_2	a_1	a_2
0.1	.050000	.050000	.050000	.050000	.050000	.050000	.050000	.050000	.050000	.050000	.050000	.050000
0.2	.100000	.100000	.100000	.100000	.100000	.100000	.100000	.100000	.100000	.100000	.100000	.100000
0.3	.150002	.149998	.150000	.150000	.150000	.150000	.150002	.149998	.150000	.150000	.150000	.150000
0.4	.200000	.200000	.200000	.200000	.200000	.200000	.200003	.199997	.200000	.200000	.200000	.200000
0.5	.250000	.250000	.250000	.250000	.250000	.250000	.250000	.250000	.250000	.250000	.250000	.250000
0.6	.300000	.300000	.300000	.300000	.300000	.300000	.300000	.300000	.300000	.300000	.300000	.300000
0.7	.350000	.350000	.350000	.350000	.350000	.350000	.350000	.350000	.350000	.350000	.350000	.350000
0.8	.400000	.400000	.400000	.400000	.400000	.400000	.400000	.400000	.400000	.400000	.400000	.400000
0.9	.450000	.450000	.450000	.450000	.450000	.450000	.450000	.450000	.450000	.450000	.450006	.449994
1.0	.500000	.500000	.500000	.500000	.500000	.500000	.500000	.500000	.500000	.500000	.500006	.499994
1.1	.550000	.550000	.550000	.550000	.550000	.550000	.550000	.550000	.550000	.550000	.550007	.549993
1.2	.600002	.599998	.600000	.600000	.600000	.600000	.600000	.600000	.600000	.600000	.600008	.599992
1.3	.650000	.650000	.650000	.650000	.650000	.650000	.650002	.649998	.650000	.650000	.650000	.650000
1.4	.700000	.700000	.700000	.700000	.700000	.700000	.700000	.700000	.700000	.700000	.700000	.700000
1.5	.370081	.370081	.597368	.598408	.750000	.750000	.750000	.750000	.750000	.750000	.750000	.750000
1.6	.229967	.229967	.319391	.321822	.800000	.800000	.491632	.491632	.800000	.800000	.800000	.800000
1.7	.151139	.151139	.206653	.20[illegible]8	.850000	.850000	.297492	.297492	.850000	.850000	.850000	.850000
1.8	.101808	.101808	.139185	.1412[illegible]4	.900000	.900000	.198331	.198331	.900000	.900000	.900000	.900000
1.9	.068973	.068973	.095073	.096782	.950000	.950000	.136104	.136104	.950000	.950000	.950000	.950000
2.0	.046635	.046635	.065140	.066488	.572487	.729990	.094366	.094366	1.000000	1.000000	1.000000	1.000000
2.1	.031318	.031318	.044493	.045526	.366495	.584137	.065525	.065525	1.050000	1.050000	1.050000	1.050000
2.2	.020819	.020819	.030173	.030945	.250267	.479946	.045329	.045329	.516767	.727038	1.100000	1.100000
2.3	.013665	.013665	.020255	.020820	.173938	.391810	.031132	.031132	.343271	.594750	1.150000	1.150000
2.4	.008839	.008839	.013431	.013835	.121306	.314226	.021171	.021171	.236537	.489596	1.200000	1.200000
2.5	.005626	.005626	.008780	.009064	.084395	.246263	.014226	.014226	.164551	.397706	1.250000	1.250000
2.6	.003519	.003519	.005652	.005846	.058427	.188332	.009430	.009430	.114474	.316081	1.300000	1.300000
2.7	.002161	.002161	.003578	.003708	.040203	.140682	.006159	.006159	.079353	.244951	1.350000	1.350000
2.8	.001302	.001302	.002225	.002311	.027471	.102851	.003959	.003959	.054753	.185148	1.400000	1.400000
2.9	.000770	.000770	.001359	.001414	.018620	.073736	.002502	.002502	.037590	.136813	1.450000	1.450000
3.0	.000446	.000446	.000815	.000849	.012503	.051912	.001554	.001554	.025659	.099106	1.500000	1.500000
3.1	.000251	.000255	.000479	.000500	.008305	.035915	.000948	.000948	.017394	.070529	1.550000	1.550000
3.2	.000140	.000142	.000276	.000289	.005450	.024423	.000568	.000568	.011692	.049371	1.600000	1.600000
3.3	.000076	.000077	.000115	.000193	.003528	.016323	.000331	.000336	.007781	.034010	1.650000	1.650000
3.4	.000041	.000041	.000064	.000107	.002251	.010719	.000191	.000194	.005118	.023056	1.700000	1.700000
3.5	.000021	.000022	.000035	.000058	.001414	.006914	.000108	.000110	.003323	.015376	1.750000	1.750000

Table 2: Bias of Maximum Probability Estimators

θ	r_1=0.5 r_2=0.5	r_1=0.5 r_2=1.5	r_1=0.5 r_2=5.0	r_1=1.5 r_2=0.5	r_1=1.5 r_2=1.5	r_1=1.5 r_2=5.0	r_1=5.0 r_2=0.5	r_1=5.0 r_2=1.5	r_1=5.0 r_2=5.0
0.0	.30462	.28916	.16534	.28950	.27224	.13703	.18708	.15778	.01186
0.1	.21134	.19583	.07088	.19618	.17885	.04212	.09391	.06330	-.08710
0.2	.13139	.11575	-.01249	.11610	.09856	-.04257	.01166	-.02014	-.18338
0.3	.06443	.04859	-.08478	.04895	.03110	-.11696	-.05882	-.09250	-.27816
0.4	.00991	-.00616	-.14599	-.00579	-.02400	-.18092	-.11768	-.15377	-.36945
0.5	-.03204	-.04921	-.19622	-.04882	-.06738	-.23434	-.16516	-.20400	-.45705
0.6	-.06506	-.08144	-.23563	-.08104	-.09987	-.27718	-.20160	-.24333	-.54005
0.7	-.08753	-.10389	-.26455	-.10348	-.12244	-.30951	-.22753	-.27202	-.61741
0.8	-.10156	-.11774	-.28346	-.11732	-.13622	-.33158	-.24364	-.29055	-.68794
0.9	-.10844	-.12423	-.29203	-.12382	-.14240	-.34384	-.25080	-.29958	-.75037
1.0	-.10946	-.12466	-.29413	-.12425	-.14228	-.34694	-.25005	-.29998	-.80342
1.1	-.10592	-.12032	-.28780	-.11992	-.13714	-.34179	-.24256	-.29281	-.84587
1.2	-.09902	-.11245	-.27523	-.11297	-.12823	-.32946	-.22961	-.27928	-.87663
1.3	-.08988	-.10218	-.25769	-.10182	-.11674	-.31119	-.21252	-.26073	-.89437
1.4	-.07946	-.09053	-.23649	-.09020	-.10373	-.28833	-.19258	-.23851	-.90007
1.5	-.06856	-.07835	-.21291	-.07805	-.09010	-.26222	-.17102	-.21395	-.89214
1.6	-.05784	-.06633	-.18816	-.06607	-.07650	-.23420	-.14893	-.18831	-.87141
1.7	-.04775	-.05498	-.16329	-.05475	-.06378	-.20550	-.12724	-.16267	-.83869
1.8	-.03860	-.04466	-.13921	-.04445	-.05206	-.17719	-.10660	-.13797	-.79524
1.9	-.03059	-.03555	-.11660	-.03538	-.04167	-.15015	-.08782	-.11491	-.74272
2.0	-.02376	-.02776	-.09598	-.02762	-.03272	-.12508	-.07097	-.09400	-.68310
2.1	-.01810	-.02126	-.07765	-.02115	-.02521	-.10242	-.05633	-.07552	-.61855
2.2	-.01353	-.01598	-.06175	-.01589	-.01906	-.08246	-.04390	-.05961	-.55130
2.3	-.00992	-.01179	-.04827	-.01172	-.01415	-.06527	-.03361	-.04621	-.48353
2.4	-.00714	-.00854	-.03710	-.00848	-.01031	-.05080	-.02528	-.03520	-.41724
2.5	-.00504	-.00607	-.02803	-.00603	-.00738	-.03887	-.01867	-.02634	-.35413
2.6	-.00350	-.00423	-.02082	-.00420	-.00518	-.02925	-.01355	-.01937	-.29559
2.7	-.00238	-.00290	-.01521	-.00288	-.00358	-.02164	-.00966	-.01399	-.24258
2.8	-.00159	-.00195	-.01092	-.00194	-.00242	-.01574	-.00677	-.00993	-.19570
2.9	-.00105	-.00129	-.00771	-.00128	-.00161	-.01126	-.00466	-.00693	-.15518
3.0	-.00067	-.00084	-.00535	-.00083	-.00106	-.00792	-.00315	-.00475	-.12092
4.0	-.00000	-.00000	-.00006	-.00000	-.00001	-.00009	-.00002	-.00004	-.00373
5.0	-.00000	-.00000	-.00000	-.00000	-.00000	-.00000	-.00000	-.00000	-.00002

Table 3: MSE of Maximum Probability Estimators

●	r_1=0.5 r_2=0.5	r_1=0.5 r_2=1.5	r_1=0.5 5_2=5.0	r_1=1.5 r_2=0.5	r_1=1.5 r_2=1.5	r_1=1.5 r_2=5.0	r_1=5.0 r_2=0.5	r_1=5.0 r_2=1.5	r_1=5.0 r_2=5.0
0.0	0.85293	0.83940	0.70630	0.83990	0.82437	0.67579	0.74356	0.71164	0.52502
0.1	0.81090	0.80040	0.69063	0.80084	0.78868	0.66526	0.72401	0.69769	0.53470
0.2	0.80138	0.79383	0.70551	0.79421	0.78531	0.68458	0.73594	0.71472	0.56785
0.3	0.81665	0.81204	0.74476	0.81236	0.80671	0.72816	0.77267	0.75654	0.62370
0.4	0.84949	0.84785	0.80252	0.84811	0.84577	0.79064	0.82782	0.81724	0.70154
0.5	0.89334	0.89472	0.87318	0.89491	0.89509	0.86684	0.89536	0.89113	0.80054
0.6	0.94245	0.94688	0.95145	0.94701	0.95158	0.95171	0.96965	0.97277	0.91957
0.7	0.99200	0.99944	1.03239	0.99950	1.00759	1.04041	1.04555	1.05702	1.05702
0.8	1.03818	1.04851	1.11150	1.04850	1.06003	1.12835	1.11850	1.13918	1.21063
0.9	1.07818	1.09118	1.18482	1.09108	1.10583	1.21133	1.18466	1.21510	1.37737
1.0	1.11023	1.12552	1.24910	1.12535	1.14295	1.28572	1.24102	1.28129	1.55337
1.1	1.13344	1.15057	1.30186	1.15033	1.17028	1.34854	1.28546	1.33515	1.73396
1.2	1.14776	1.16619	1.34152	1.16580	1.18757	1.39766	1.31680	1.37497	1.91370
1.3	1.15379	1.17292	1.39737	1.17257	1.19520	1.43185	1.33479	1.40002	2.08664
1.4	1.15260	1.17183	1.37955	1.17145	1.19450	1.45081	1.34004	1.41050	2.24655
1.5	1.14557	1.16435	1.37901	1.16395	1.18665	1.45509	1.33383	1.40745	2.38731
1.6	1.13421	1.15205	1.36729	1.15165	1.17339	1.44605	1.31799	1.39260	2.50331
1.7	1.12003	1.13654	1.34638	1.13614	1.156[illegible]2	1.42562	1.29467	1.36816	2.58987
1.8	1.10439	1.11928	1.31854	1.11891	1.13734	1.39614	1.26613	1.33660	2.64362
1.9	1.08846	1.10157	1.28608	1.10122	1.11757	1.36016	1.23460	1.30045	2.66280
2.0	1.07312	1.08439	1.25118	1.08408	1.09825	1.32020	1.20205	1.26208	2.64740
2.1	1.05903	1.06851	1.21580	1.06823	1.08022	1.27861	1.17016	1.22359	2.59921
2.2	1.04659	1.05437	1.18155	1.05413	1.06406	1.23740	1.14021	1.18667	2.52163
2.3	1.03596	1.04221	1.14963	1.04201	1.05005	1.19821	1.11311	1.15250	2.41946
2.4	1.02716	1.03208	1.12087	1.03192	1.03828	1.16223	1.08937	1.12218	2.29841
2.5	1.02009	1.02387	1.09574	1.02374	1.02867	1.13022	1.06918	1.09586	2.16475
2.6	1.01455	1.01740	1.07438	1.01730	1.02104	1.10250	1.05248	1.07371	2.02473
2.7	1.01032	1.01243	1.05669	1.01235	1.01514	1.07917	1.03903	1.05557	1.88424
2.8	1.00718	1.00870	1.04240	1.00864	1.01067	1.06901	1.02846	1.04108	1.74837
2.9	1.00489	1.00597	1.03113	1.00593	1.00738	1.04465	1.02036	1.02978	1.62124
3.0	1.00327	1.00402	1.02244	1.00399	1.00500	1.03262	1.01428	1.02118	1.50582
4.0	1.00002	1.00003	1.00032	1.00003	1.00004	1.00053	1.00015	1.00025	1.02231
5.0	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00014

Table 4: Values of $b(m) = \max\limits_{\omega=0.0(0.1)5.0} |B^{*(m)}(\omega)|$, the maximum |bias| of m-th IBEE.

m	b(m)	m	b(m)	m	b(m)
1	0.5642	2	0.3305	3	0.2761
4	0.2499	5	0.2338	6	0.2227
7	0.2144	8	0.2079	9	0.2027
10	0.1983	11	0.1945	12	0.1913
13	0.1885	14	0.1860	15	0.1832
16	0.1840	17	0.1822	18	0.1536
19	0.1879	20	0.2736	21	0.2047
22	1.4915	23	0.3366	24	4.3284
25	1.0571	26	16.4563	27	4.1254
28	105.9889	29	23.4225	30	409.0409

Table 5: Bias Comparison

ω	$\delta_H(1)$	δ_P	$t_2(0.5,0.5)$	$\chi^{*}(18)$
0.0	0.2076	0.7979	0.3046	0.1536
0.1	0.1138	0.7019	0.2113	0.0669
0.2	0.0324	0.6138	0.1314	0.0016
0.3	-0.0367	0.5335	0.0644	-0.0389
0.4	-0.0938	0.4609	0.0099	-0.0655
0.5	-0.1392	0.3956	-0.0329	-0.0734
0.6	-0.1734	0.3374	-0.0651	-0.0746
0.7	-0.1972	0.2858	-0.0875	-0.0621
0.8	-0.2114	0.2404	-0.1016	-0.0446
0.9	-0.2169	0.2009	-0.1084	-0.0280
1.0	-0.2151	0.1667	-0.1095	-0.0219
1.1	-0.2071	0.1372	-0.1059	-0.0062
1.2	-0.1942	0.1122	-0.0990	0.0016
1.3	-0.1779	0.0911	-0.0899	0.0126
1.4	-0.1593	0.0733	-0.0795	0.0016
1.5	-0.1397	0.0586	-0.0686	0.0187
1.6	-0.1200	0.0465	-0.0578	0.0038
1.7	-0.1010	0.0366	-0.0478	0.0059
1.8	-0.0833	0.0286	-0.0386	0.0004
1.9	-0.0674	0.0221	-0.0306	-0.0010
2.0	-0.0535	0.0170	-0.0238	-0.0109
2.1	-0.0417	0.0129	-0.0181	-0.0057
2.2	-0.0318	0.0098	-0.0135	-0.0026
2.3	-0.0238	0.0073	-0.0100	-0.0228
2.4	-0.0175	0.0054	-0.0071	-0.0177
2.5	-0.0125	0.0040	-0.0050	-0.0104
2.6	-0.0089	0.0029	-0.0035	-0.0003
2.7	-0.0062	0.0021	-0.0024	-0.0076
2.8	-0.0042	0.0015	-0.0016	-0.0139
2.9	-0.0028	0.0011	-0.0011	-0.0024
3.0	-0.0019	0.0008	-0.0007	-0.0174
4.0	-0.0000	0.0000	-0.0000	-0.0015
5.0	0.0000	0.0000	-0.0000	0.0004

Table 6: MSE Comparison

ω	$\delta_H(1)$	δ_P	$t_2(0.5,0.5)$	$\chi^{*}(18)$
0.0	0.7862	1.2182	0.8529	3.0538
0.1	0.7633	1.0455	0.8109	3.0725
0.2	0.7729	0.9067	0.8014	3.0582
0.3	0.8076	0.7970	0.8167	2.9850
0.4	0.8609	0.7113	0.8495	2.9157
0.5	0.9261	0.6453	0.8933	2.7558
0.6	0.9974	0.5989	0.9425	2.6158
0.7	1.0694	0.5674	0.9920	2.4251
0.8	1.1375	0.5520	1.0382	2.2187
0.9	1.1980	0.5496	1.0782	2.0066
1.0	1.2479	0.5590	1.1102	1.8548
1.1	1.2854	0.5795	1.1334	1.7210
1.2	1.3098	0.6070	1.1478	1.5996
1.3	1.3212	0.6417	1.1538	1.4964
1.4	1.3205	0.6794	1.1526	1.4021
1.5	1.3093	0.7189	1.1456	1.2870
1.6	1.2897	0.7583	1.1342	1.1914
1.7	1.2639	0.7956	1.1200	1.1325
1.8	1.2342	0.8305	1.1044	1.0869
1.9	1.2028	0.8615	1.0885	1.0673
2.0	1.1715	0.8888	1.0731	1.0432
2.1	1.1417	0.9120	1.0590	1.0229
2.2	1.1144	0.9315	1.0466	1.0163
2.3	1.0904	0.4474	1.0360	1.0130
2.4	1.0699	0.9601	1.0272	1.0169
2.5	1.0529	0.9702	1.0201	1.0194
2.6	1.0391	0.9780	1.0146	1.0207
2.7	1.0284	0.9840	1.0103	1.0213
2.8	1.0202	0.9885	1.0072	1.0206
2.9	1.0140	0.9918	1.0049	1.0185
3.0	1.0096	0.9942	1.0033	1.0155
4.0	1.0001	0.9999	1.0000	0.9979
5.0	1.0000	1.0000	1.0000	1.0024

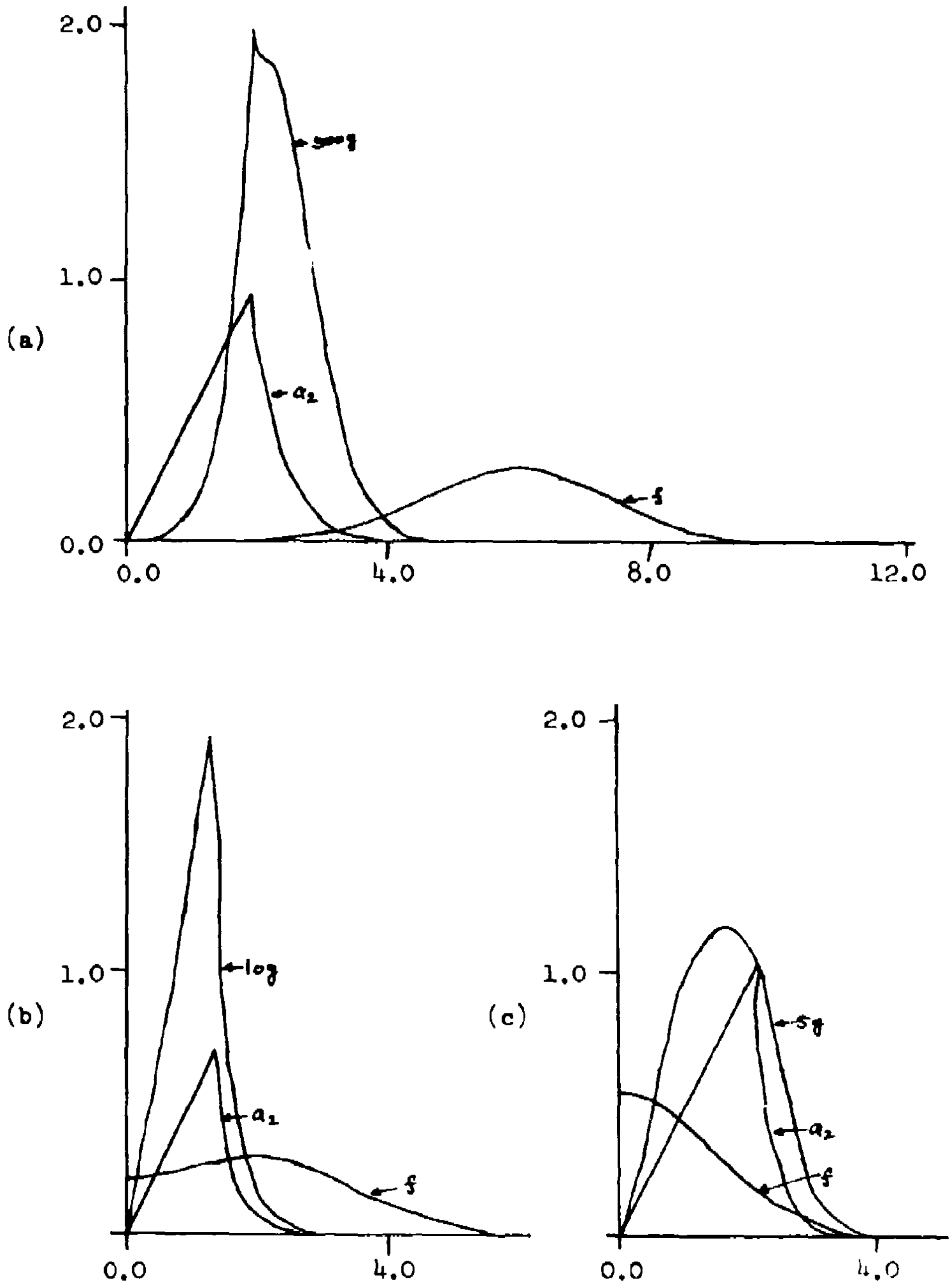

Figure 1: Graphical investigation of behavior of MPE's. f, the density of Z; a_2, the adjustment in $\bar{X}_{[2]}$ given by MPE of $\mu_{[2]}$; and $g = a_2 f$ are plotted against Z. For all (r_1, r_2) pairs considered, (a), (b), (c) represent the typical plots when ω = 3.0, 1.0, 0.0 respectively.

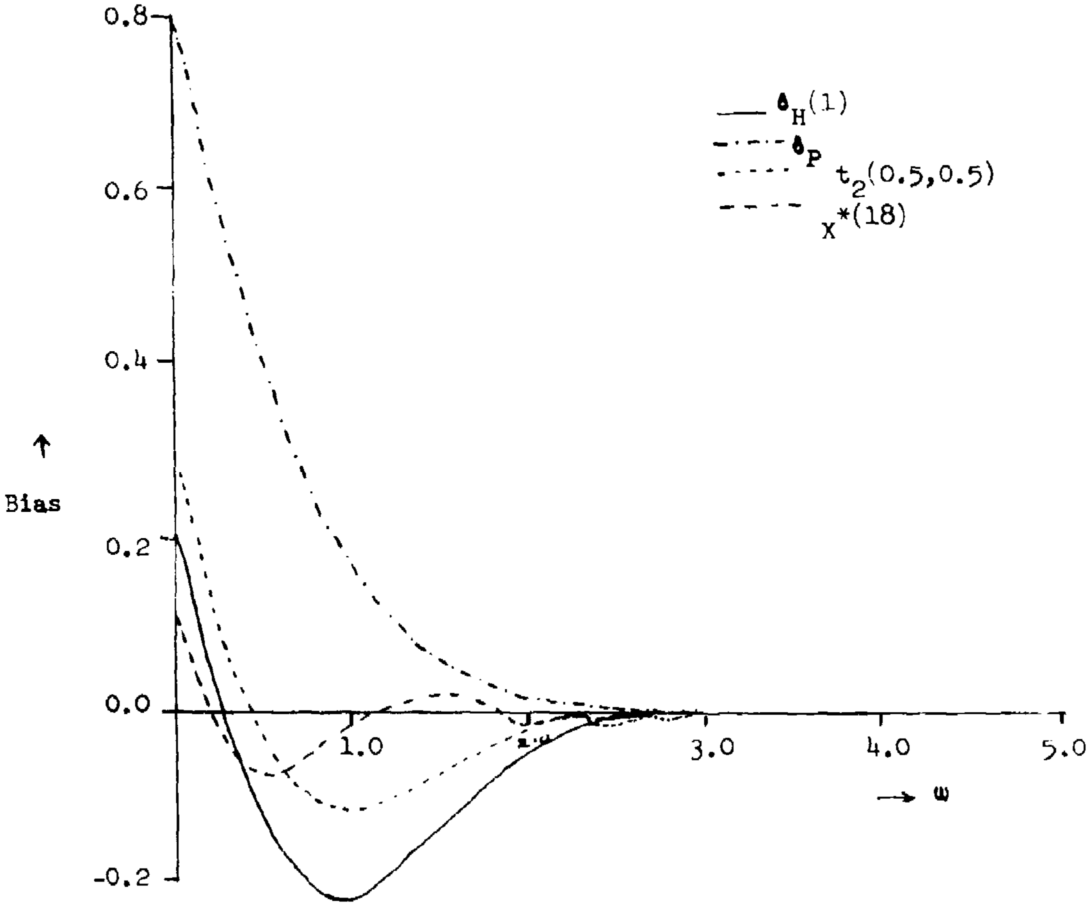

Figure 2: Comparison of estimators of $\mu_{[2]}$ with respect to their bias.

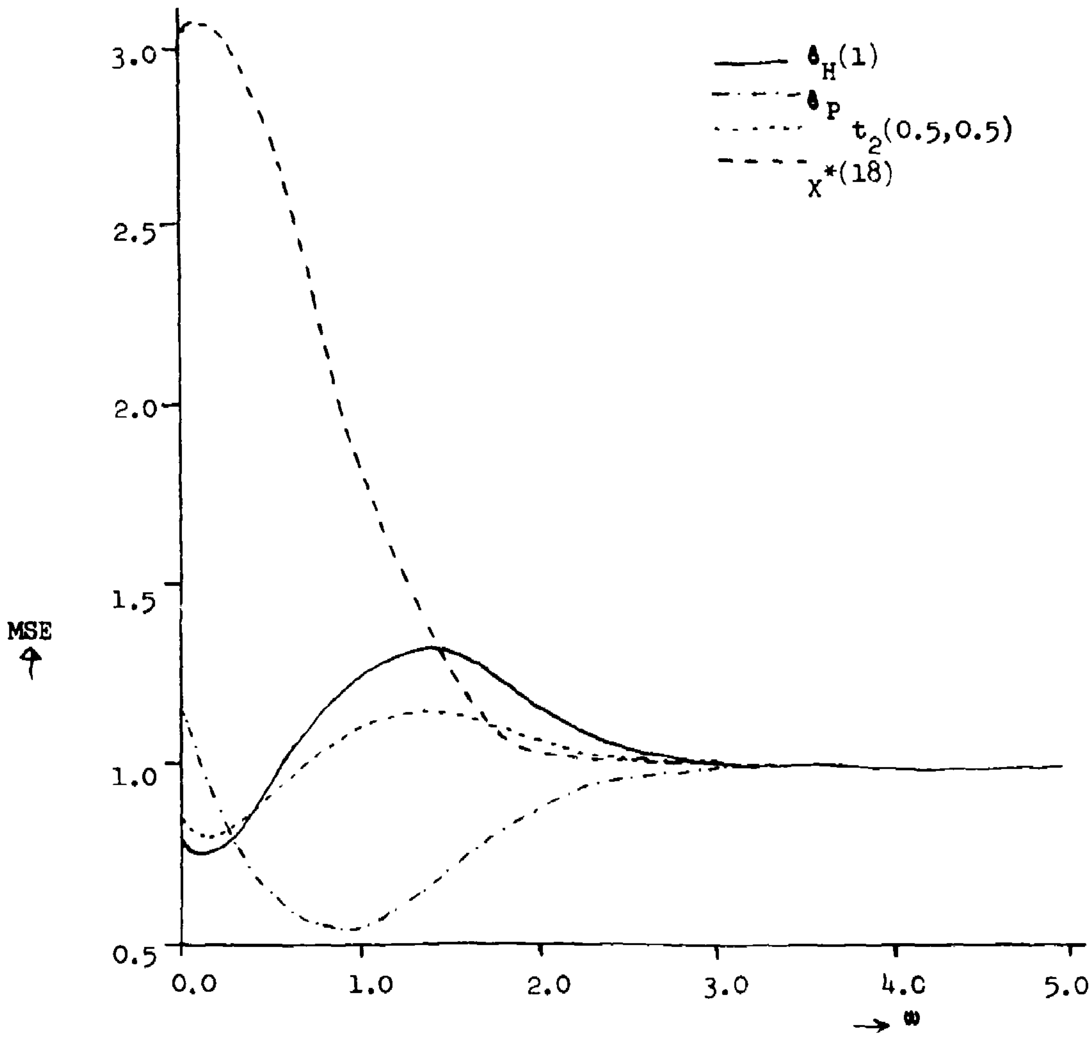

Figure 3: Comparison of estimators of $\mu_{[2]}$ with respect to their MSE.

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